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On a Class of Right Alternative Rings without Nilpotent Ideals

MARGARET M. HUMM[1]

Department of Mathematics, Syracuse University, Syracuse, New York

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1. INTRODUCTION

Let A be a ring. If x, y, and z are elements of A we define the associator $(x, y, z) = xy \cdot z - x \cdot yz$ and the commutator $(x, y) = xy - yx$. A right alternative ring is a ring R which satisfies the identity $(x, y, y) = 0$ for all elements x and y or R. A right alternative ring R of characteristic not 2 has been shown to be alternative under any one of the following hypotheses:

(a) (Albert) R is a simple finite-dimensional algebra [4];
(b) (Skornyakov) R is a division ring [7];
(c) (Kleinfeld) R is a ring without nilpotent elements [6].

In this paper a necessary and sufficient condition is found for a simple right alternative ring to be alternative. We assume characteristic not 2 or 3 in all that follows. Our treatment requires an idempotent e in R and uses the subspaces $R_1(e)$ and $R_0(e)$ of the Albert decomposition [1].

THEOREM. *A simple right alternative ring R which is not associative is alternative if and only if R has an idempotent e such that there are no nilpotent elements in $R_1(e)$ and $R_0(e)$.*

Because simple alternative rings that are not associative are known to be Cayley–Dickson algebras [5] one easily sees that the condition is necessary. Most of the paper is taken up with establishing that it is also sufficient. In the process we prove the following more general result.

THEOREM. *A right alternative ring R with no proper nilpotent ideals and having an idempotent e for which there are no nilpotent elements in $R_1(e)$ and $R_0(e)$ is alternative.*

[1] This paper is from the author's doctoral dissertation which was done under the guidance of Erwin Kleinfeld.

This subsumes the result of Kleinfeld [6]. To see this, note that an idempotent may be adjoined to R if one is not already present by forming the direct product of R with a field of characteristic not 2 or 3. The resulting ring will satisfy our hypotheses if R has no nilpotent elements. In proving our theorem we first show that R must have a Pierce decomposition if there are no nilpotent elements in $R_1(e)$ and $R_0(e)$. Using this assumption and some identities we are able to obtain quite a bit of information about the multiplication of subspaces. We then construct the set $S = S_{10} + S_{01}$ where S_{ij} is the set of all elements in R_{ij} which annihilate R_{ji} on both sides. S turns out to be an ideal whose cube is zero. Using this ideal we are able to show that if R has no nilpotent ideals the multiplication of subspaces is exactly the same as in the case of an alternative ring. Some further computations show that R is alternative.

2. Preliminaries

Let R be a right alternative ring and let x, y, z, and w be elements of R. The following results through Lemma 4 may be found in [6]. Some of them are due originally to Skornyakov.

$$(x, y, z) = -(x, z, y). \tag{1}$$

$$(x, y, yz) = (x, y, z)y. \tag{2}$$

The following functions vanish identically on R:

$$\begin{aligned} f(w, x, y, z) &= (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z; \\ g(x, w, y, z) &= (x, w, yz) + (x, y, wz) - (x, w, z)y - (x, y, z)w; \\ h(w, x, y, z) &= (wx, y, z) + (w, x, (y, z)) - w(x, y, z) - (w, y, z)x; \\ k(x, y, z) &= (x, y^2, z) - (x, y, yz + zy). \end{aligned}$$

LEMMA 1. $(xy \cdot z)y = x(yz \cdot y)$.

DEFINITION. *For fixed elements a, b in R we define $S(a, b)$ as the set of all elements x in R which have the property that $(x, a, b) = x(b, a)$.*

LEMMA 2. *If x belongs to both $S(a, b)$ and $S(a, ba)$, then $x(a, a, b) = 0$.*

LEMMA 3. *If both y and xy belong to $S(a, b)$, then $(x, a, b)y = 0$.*

LEMMA 4. *For arbitrary x, (x, a, b) and $(x, a, b)a$ are both contained in $S(a, b)$.*

Assume that R has an idempotent e. Since R is power associative [2] we may form the Albert decomposition [1] $R = R_1(e) + R_{1/2}(e) + R_0(e)$ where $R_i(e)$ is the set of all x in R such that $ex + xe = 2ix$. It has been established by Albert [1] that $R_1(e)R_0(e) = R_0(e)R_1(e) = 0$ and $ex_1 = x_1e = x_1$ and $ex_0 = x_0e = 0$ for elements x_1 in $R_1(e)$ and x_0 in $R_0(e)$.

3. Main Section

It has been shown [2] that, in general, a right alternative ring need not have a Pierce decomposition; however we have the following:

Lemma 5. *If R is a right alternative ring with an idempotent e such that there are no nilpotent elements in $R_1(e)$ and $R_0(e)$, then $(e, e, x) = 0$ for all x in R, and thus R has a Pierce decomposition.*

Proof. From $f(e, e, e, x) = 0$ we get $e(e, e, x) = (e, e, ex)$, but from (2) we get $(e, e, ex) = (e, e, x)e$. If we let $t = (e, e, x)$ we have $(e, t) = 0$. By Lemma 4 we have that t is in $S(e, x)$ and te is in $S(e, x)$, thus et is in $S(e, x)$. Now by Lemma 3 with $y = t$, $x = e$, $a = e$, and $b = x$, we have $t^2 = 0$. Now $t = t_1 + t_{1/2} + t_0$ with t_i in $R_i(e)$. Since e acts as an identity on $R_1(e)$ and annihilates $R_0(e)$ we have $(R_1(e), e) = (R_0(e), e) = 0$; thus $(t_{1/2}, e) = 0$ and hence $t_{1/2}e = \frac{1}{2}t_{1/2}$. Now $t_{1/2}e \cdot e = t_{1/2}e^2 = t_{1/2}e$ implies $\frac{1}{4}t_{1/2} = \frac{1}{2}t_{1/2}$; hence $t_{1/2} = 0$. We have $0 = t^2 = t_1^2 + t_0^2$. Now x_i^2 is in $R_i(e)$ for all x_i in $R_i(e)$, $i = 0$, or 1, because $ex_i^2 = ex_i \cdot x_i = ix_i^2$ and $(x_i, e, x_i) = 0$ implies $(x_i, x_i, e) = 0$ which gives $x_i^2e = x_i \cdot x_ie = ix_i^2$. Thus $t_1^2 + t_0^2 = 0$ implies $t_1^2 = 0$ and $t_0^2 = 0$ since $R_1(e)$ and $R_0(e)$ are direct summands. Now by assumption there are no nilpotent elements in $R_1(e)$ or $R_0(e)$, so we conclude that $t_1 = t_0 = 0$; hence $t = 0$. We now have a Pierce direct sum decompositon of R. $R = R_{11} + R_{10} + R_{01} + R_{00}$ where R_{ij} is the set of all x in R for which $ex = ix$ and $xe = jx$. We shall write x_{ij} for a generic element of R_{ij}, etc., throughout the remainder of the paper.

Lemma 6. *Under the assumptions of Lemma 5 we have the following table for the multiplication of subspaces.*

	R_{11}	R_{10}	R_{01}	R_{00}
R_{11}	$R_{11} + R_{01}$	R_{10}	R_{10}	0
R_{10}	0	$R_{11} + R_{01}$	R_{11}	R_{10}
R_{01}	R_{01}	R_{00}	$R_{00} + R_{10}$	0
R_{00}	0	R_{01}	R_{01}	$R_{00} + R_{10}$

The table is to be read as follows: the subspace in column 1 *row s multiplied on the right by the subspace in row* 1 *column t is contained in the subspace listed in row s column t. Moreover,* x_{ii}^2 *is in* R_{ii} *and* x_{ij}^2 *is in* R_{ii}.

Proof. In the rest of the paper we assume $i \neq j$. We note that when the Pierce decompositon exists $R_{ii} = R_i(e)$ for $i = 0, 1$; thus $x_{ii}^2 \in R_{ii}$ has been established in the proof of Lemma 5. Expanding $(x_{ij}, e, y_{ij}) + (x_{ij}, y_{ij}, e) = 0$ gives $x_{ij}y_{ij} \cdot e = ix_{ij}y_{ij}$; thus $x_{ij}y_{ij}$ is in $R_{ii} + R_{ji}$. Since $ex_{ij}^2 = ex_{ij} \cdot x_{ij} = ix_{ij}^2$ we have shown that x_{ij}^2 is in R_{ii}. By (1) we have $(x_{ij}, y_{ji}, e) = -(x_{ij}, e, y_{ji}) = 0$ and expanding gives $x_{ij}y_{ji} \cdot e = ix_{ij}y_{ji}$. From $g(e, x_{ij}, y_{ji}, e) = 0$ we obtain $(i-j)(e, x_{ij}, y_{ji}) = 0$. Thus $(e, x_{ij}, y_{ji}) = 0$, and expanding gives $e \cdot x_{ij}y_{ji} = ix_{ij}y_{ji}$ so that $x_{ij}y_{ji}$ is in R_{ii}. From (1) we have $(y_{ii}, x_{ij}, e) = -(y_{ii}, e, x_{ij}) = 0$ and expanding gives $y_{ii}x_{ij} \cdot e = jy_{ii}x_{ij}$. From $g(e, x_{ij}, y_{ii}, e) = 0$ we obtain $(j - i)(e, y_{ii}, x_{ij}) = 0$, whence $(e, y_{ii}, x_{ij}) = 0$. Expanding the last associator gives $e \cdot y_{ii}x_{ij} = iy_{ii}x_{ij}$ and thus $y_{ii}x_{ij}$ belongs to R_{ij}. From $k(x_{ij}, e, y_{ii}) = 0$ we obtain $(1 - 2i)(x_{ij}, e, y_{ii}) = 0$. Since i is either 1 or 0, we conclude that $(x_{ij}, e, y_{ii}) = 0$. Expanding this gives $(j - i)(x_{ij}y_{ii}) = 0$ and thus $x_{ij}y_{ii} = 0$. We have $(y_{10}, x_{00}, e) = -(y_{10}, e, x_{00})$, and expanding gives $y_{10}x_{00} \cdot e = 0$. By (1) we have $0 = (e, x_{00}, y_{10}) + (e, y_{10}, x_{00})$. Expanding gives

$$0 = -e \cdot x_{00}y_{10} + y_{10}x_{00} - e \cdot y_{10}x_{00}. \tag{3}$$

Multiplying (3) on the left by e yields $0 = -e \cdot x_{00}y_{10}$. We substitute this information in (3) to obtain $e \cdot y_{10}x_{00} = y_{10}x_{00}$. Thus $y_{10}x_{00}$ is in R_{10}. Now $0 = (x_{00}, e, y_{10}) + (x_{00}, y_{10}, e) = -x_{00}y_{10} + x_{00}y_{10} \cdot e$; hence $x_{00}y_{10} \cdot e = x_{00}y_{10}$. We have already seen that $e \cdot x_{00}y_{10} = 0$, thus $x_{00}y_{10}$ is in R_{01}. From (1) we have $(x_{01}, y_{11}, e) = -(x_{01}, e, y_{11}) = 0$ and expanding gives $x_{01}y_{11} \cdot e = x_{01}y_{11}$. From (1) we have $0 = (e, x_{01}, y_{11}) + (e, y_{11}, x_{01})$. Expanding gives

$$-e \cdot x_{01}y_{11} + y_{11}x_{01} - e \cdot y_{11}x_{01} = 0. \tag{4}$$

Multiplying (4) on the left by e gives $-e \cdot x_{01}y_{11} = 0$. We have shown that $x_{01}y_{11}$ is in R_{01}. Substituting $e \cdot x_{01}y_{11} = 0$ in (4) gives $y_{11}x_{01} = e \cdot y_{11}x_{01}$, while expanding $(y_{11}, e, x_{01}) + (y_{11}, x_{01}, e) = 0$ gives $0 = y_{11}x_{01} + y_{11}x_{01} \cdot e - y_{11}x_{01}$. Thus $y_{11}x_{01} \cdot e = 0$ and we have that $y_{11}x_{01}$ is in R_{10}. From (1) we have $(x_{ii}, y_{ii}, e) = -(x_{ii}, e, y_{ii}) = 0$. Expanding gives $x_{ii}y_{ii} = ix_{ii}y_{ii}$, hence $x_{ii}y_{ii}$ belongs to $R_{ii} + R_{ji}$.

DEFINITION. *A triple of elements* $x, y,$ *and* z *is called an alternative triple if* $(\sigma(x), \sigma(y), \sigma(z)) = \operatorname{sgn} \sigma(x, y, z)$ *for all permutations* σ *of* x, y, *and* z *within the associator.*

LEMMA 7. *If* R *is as above, then* $x_{ij}^2 = 0$ *for all* x_{ij} *in* R_{ij} *and thus* R_{ij} *is anticommutative. Moreover,* $(x_{ij}, x_{ij}, y_{ij}) = 0$, *and thus every three elements of* R_{ij} *form an alternative triple.*

Proof. By Lemma 6 we know that x_{ij}^2 is in R_{ii} and thus $x_{ij} \cdot x_{ij}^2 = 0$. It follows that both x_{ij}^3 and x_{ij}^4 are equal to zero. Since there are no nilpotent elements in R_{ii} it follows that $x_{ij}^2 = 0$. Substituting $y_{ij} + z_{ij}$ for x_{ij} in the previous equation gives $y_{ij}z_{ij} = -z_{ij}y_{ij}$. Now expanding and using $x_{ij}^2 = 0$ gives $(x_{ij}, x_{ij}, y_{ij}) = -x_{ij} \cdot x_{ij}y_{ij} \in R_{ij}(R_{ji} + R_{ii}) \subset R_{ii}$. Since $R_{ij}R_{ii} = 0$ we have $x_{ij}(x_{ij}, x_{ij}, y_{ij}) = 0$, thus $x_{ij}(x_{ij}, x_{ij}, y_{ij})$ is in $S(x_{ij}, y_{ij})$. By Lemma 4 we know that (x_{ij}, x_{ij}, y_{ij}) is in $S(x_{ij}, y_{ij})$. Applying Lemma 3 with $x = x_{ij}, y = (x_{ij}, x_{ij}, y_{ij})$ gives $(x_{ij}, x_{ij}, y_{ij})^2 = 0$. Since (x_{ij}, x_{ij}, y_{ij}) is in R_{ii} we conclude that $(x_{ij}, x_{ij}, y_{ij}) = 0$. Replacing x_{ij} by $w_{ij} + z_{ij}$ in this equation gives $(w_{ij}, z_{ij}, y_{ij}) = -(z_{ij}, w_{ij}, y_{ij})$, and this together with (1) is enough to make every three elements of R_{ij} an alternative triple.

LEMMA 8. *If R is as above, then $a_{ij}b_{ji} = 0$ implies $b_{ji}a_{ij} = 0$.*

Proof. We have $(a_{ij}, b_{ji}, a_{ij}) = -(a_{ij}, a_{ij}, b_{ji}) = a_{ij}^2 b_{ji} + a_{ij} \cdot a_{ij}b_{ji}$. But $a_{ij}^2 = 0$ by Lemma 7, and $a_{ij} \cdot a_{ij}b_{ji} = 0$ by the hypothesis. Thus

$$(a_{ij}, b_{ji}, a_{ij}) = 0. \tag{5}$$

We also have $0 = k(b_{ji}, a_{ij}, b_{ji}) = (b_{ji}, a_{ij}, b_{ji}a_{ij})$, since the other terms vanish; thus

$$(b_{ji}, a_{ij}, b_{ji}a_{ij}) = 0. \tag{6}$$

Now $b_{ji}a_{ij} \cdot b_{ji}a_{ij} = b_{ji}(a_{ij} \cdot b_{ji}a_{ij})$ because of (6) and $a_{ij} \cdot b_{ji}a_{ij} = a_{ij}b_{ji} \cdot a_{ij} = 0$ because of (5) and the hypothesis. These last two equations imply $(b_{ji}a_{ij})^2 = 0$, and since $b_{ji}a_{ij}$ is in R_{jj}, this implies $b_{ji}a_{ij} = 0$.

LEMMA 9. *If R is as above, the following associators vanish:*

$$(R_{ii}, R_{ij}, R_{ii}) = (R_{ii}, R_{ii}, R_{ij}) = 0, \tag{7}$$

$$(R_{ii}, R_{jj}, R_{ii}) = (R_{ii}, R_{ii}, R_{jj}) = 0, \tag{8}$$

$$(R_{ij}, R_{ii}, R_{jj}) = (R_{ij}, R_{jj}, R_{ii}) = 0, \tag{9}$$

$$(R_{ji}, R_{ij}, R_{ii}) = (R_{ji}, R_{ii}, R_{ij}) = 0, \tag{10}$$

$$(R_{ij}, R_{ii}, R_{ii}) = 0. \tag{11}$$

Proof. In (7), (8), (9), and (10) the first associator can be seen to vanish by expanding and using Lemma 6. The second associator in these equations is the negative of the first by (1) and hence vanishes also. To establish (11) note that by expanding and using Lemma 6 we have $(R_{ij}, R_{ii}, R_{ii}) \subset R_{ii}$. Thus we have $x_{ij}(x_{ij}, y_{ii}, z_{ii}) = 0$ by Lemma 6, and (x_{ij}, y_{ii}, z_{ii}) is in $S(y_{ii}, z_{ii})$ by Lemma 4. Taking $x = x_{ij}$, $y = (x_{ij}, y_{ii}, z_{ii})$ in Lemma 3 we see that $(x_{ij}, y_{ii}, z_{ii})^2 = 0$; hence (x_{ij}, y_{ii}, z_{ii}) in R_{ii} implies $(x_{ij}, y_{ii}, z_{ii}) = 0$.

LEMMA 10. *If R is as above, then* $R_{ij}R_{ij} \subset R_{ji}$.

Proof. By Lemma 6 we know that $x_{ij}y_{ij}$ is an element of $R_{ji} + R_{ii}$, so we may write $x_{ij}y_{ij} = w_{ji} + r_{ii}$ for some w_{ji} in R_{ji} and r_{ii} in R_{ii}. Consider $w_{ji}r_{ii} + r_{ii}^2 = (w_{ji} + r_{ii})r_{ii} = x_{ij}y_{ij} \cdot r_{ii} = (x_{ij}, y_{ij}, r_{ii}) = -(x_{ij}, r_{ii}, y_{ij}) = x_{ij} \cdot r_{ii}y_{ij}$. We have used the definition of the associator, Lemma 6, and (1). Thus we have

$$w_{ji}r_{ii} + r_{ii}^2 = x_{ij} \cdot r_{ii}y_{ij}. \tag{12}$$

Now we also have $w_{ji}y_{ij} + r_{ii}y_{ij} = (w_{ji} + r_{ii})y_{ij} = x_{ij}y_{ij} \cdot y_{ij} = x_{ij}y_{ij}^2 = 0$. Since by Lemma 6 we have $w_{ji}y_{ij}$ is in R_{jj} and $r_{ii}y_{ij}$ is in R_{ij}, and since R_{jj} and R_{ij} are direct summands, we conclude that $r_{ii}y_{ij} = 0$. Combining this with (12) gives $w_{ji}r_{ii} + r_{ii}^2 = 0$. Since $w_{ji}r_{ii}$ is in R_{ji} and r_{ii}^2 is in R_{ii}, we conclude that $r_{ii}^2 = 0$; hence $r_{ii} = 0$. Thus $x_{ij}y_{ij} = w_{ji}$.

LEMMA 11. *If R is as above, and if x, y, and z are elements of* R_{ij}, *then* $\sigma(x)\sigma(y) \cdot \sigma(z) = \operatorname{sgn} \sigma(xy \cdot z)$ *and* $\sigma(x) \cdot \sigma(y)\sigma(z) = \operatorname{sgn} \sigma(x \cdot yz)$.

Proof. Let x, w, y, and z be elements of R_{ij}; then by (1) we have $xw \cdot w = xw^2$. By Lemma 7 we have $w^2 = 0$. Hence $xw \cdot w = 0$. Substituting $w = y + z$ gives $xy \cdot z = -xz \cdot y$. Since R_{ij} is anticommutative, $xy \cdot z = -yx \cdot z$ and $x \cdot yz = -x \cdot zy$. It is proved in Lemma 7 that $(w, w, z) = 0$; thus $w \cdot wz = w^2 z = 0$. Substituting $w = x + y$ in $w \cdot wz = 0$ gives $x \cdot yz = -y \cdot xz$.

THEOREM 1. *Let R be a right alternative ring with an idempotent e such that there are no nilpotent elements in* $R_1(e)$ *and* $R_0(e)$. *Let* S_{ij} *be the set of all elements in* R_{ij} *which annihilate* R_{ji} *on the left. Then the set* $S = S_{10} + S_{01}$ *is an ideal of R and* $S^3 = 0$.

Proof. We note that because of Lemma 8 an arbitrary element y of R_{ij} is in S_{ij} if and only if $yR_{ji} = R_{ji}y = 0$, and to show that y is in S_{ij} it is sufficient to show either $yR_{ji} = 0$ or $R_{ji}y = 0$. Let s_{ij} be an element of S_{ij}. We have $r_{ii}s_{ij}$ is in R_{ij} by Lemma 6. Lemma 9, Eq. (10) gives $(x_{ji}, r_{ii}, s_{ij}) = 0$. It follows that $x_{ji} \cdot r_{ii}s_{ij} = x_{ji}r_{ii} \cdot s_{ij} = 0$, hence $r_{ii}s_{ij}$ is in S_{ij}. By Lemma 10 we have $r_{ij}s_{ij}$ is in R_{ji} and by Lemma 11 we have $r_{ij}s_{ij} \cdot x_{ij} = -r_{ij}x_{ij} \cdot s_{ij} = 0$. Thus $r_{ij}s_{ij}$ is in S_{ji}. By the definition of S_{ij} and Lemma 8 we have $R_{ji}S_{ij} = 0$. By Lemma 6 we have $r_{jj}s_{ij}$ is in R_{ji}. We may write $r_{jj}s_{ij} \cdot x_{ij} = (r_{jj}, s_{ij}, x_{ij}) + r_{jj} \cdot s_{ij}x_{ij} = -(r_{jj}, x_{ij}, s_{ij}) + r_{jj} \cdot s_{ij}x_{ij} = -r_{jj}x_{ij} \cdot s_{ij} + r_{jj} \cdot x_{ij}s_{ij} +$

$r_{jj} \cdot s_{ij}x_{ij}$. Now, of the three terms at the end of this equation, the first is zero by Lemma 6 and the sum of the other two is zero because R_{ij} is anticommutative. We conclude that $r_{jj}s_{ij} \cdot x_{ij} = 0$ and hence $r_{jj}s_{ij}$ is in S_{ji}. We have established that $RS_{ij} \subset S$. Now we will show that $S_{ij}R \subset S$. By Lemma 6 we have that $s_{ij}r_{jj}$ is in R_{ij}. By Lemma 9, Eq. (10) we have $s_{ij}r_{jj} \cdot x_{ji} = s_{ij} \cdot r_{jj}x_{ji} = 0$, since $r_{jj}x_{ji}$ is in R_{ji} (Lemma 6). Thus $s_{ij}r_{jj}$ is in S_{ij}. To show that $S^3 = 0$ let s_{ij}, t_{ij}, and r_{ij} be elements of S_{ij} and let m_{ji} be an element of S_{ji}. By Lemma 10 we have $s_{ij}t_{ij}$ is in R_{ji}. Thus since r_{ij} is in S_{ij} we have

$$s_{ij}t_{ij} \cdot r_{ij} = r_{ij} \cdot s_{ij}t_{ij} = 0. \tag{13}$$

We also have $s_{ij}t_{ij} \cdot m_{ji} = (s_{ij}, t_{ij}, m_{ji}) = -(s_{ij}, m_{ji}, t_{ij}) = 0$. This and the fact that R_{ji} is anticommutative imply

$$m_{ji} \cdot s_{ij}t_{ij} = -s_{ij}t_{ij} \cdot m_{ji} = 0. \tag{14}$$

Let r, s, and t be elements of S and let $r = r_{10} + r_{01}$, $s = s_{10} + s_{01}$, and and $t = t_{10} + t_{01}$. Using $S_{ij}S_{ji} = S_{ji}S_{ij} = 0$, Eq. (13), and Eq. (14) one can see that $rs \cdot t = 0$ and $r \cdot st = 0$ since each term in the expansions of these products is zero.

LEMMA 12. *If R is a right alternative ring having an idempotent e such that there are no nilpotent elements in $R_1(e)$ and $R_0(e)$ and having no proper nilpotent ideals, then $R_{ii}R_{ji} = 0$.*

Proof. Let S be as defined in Theorem 1. Since S is nilpotent, e is not in S; hence $S = 0$. We will show that $R_{ii}R_{ji} \in S$. Let x_{ii} be an element of R_{ii}, and let y_{ji} and z_{ji} be elements of R_{ji}. We have $x_{ii}y_{ji}$ is in R_{ij} by Lemma 6. Let z_{ji} be any element of R_{ji}. We have

$$\begin{aligned} z_{ji} \cdot x_{ii}y_{ji} &= -(z_{ji}, x_{ii}, y_{ji}) + z_{ji}x_{ii} \cdot y_{ji} = (z_{ji}, y_{ji}, x_{ii}) + z_{ji}x_{ii} \cdot y_{ji} \\ &= z_{ji}y_{ji} \cdot x_{ii} - z_{ji} \cdot y_{ji}x_{ii} + z_{ji}x_{ii} \cdot y_{ji}. \end{aligned}$$

Using Lemmas 6 and 10 we see that $z_{ji}y_{ji} \cdot x_{ii} = 0$ so that the preceding equation gives $z_{ji} \cdot x_{ii}y_{ji} = -z_{ji} \cdot y_{ji}x_{ii} + z_{ji}x_{ii} \cdot y_{ji}$. By Lemma 6 the left hand side of this equation is in R_{jj} while the right hand side is in R_{ij}, thus we must have $z_{ji} \cdot x_{ii}y_{ji} = 0$. We have shown that $x_{ii}y_{ji}$ is in $S_{ij} \subset S$ and therefore $x_{ii}y_{ji} = 0$.

LEMMA 13. *If R satisfies the hypotheses of Lemma 12 then the multiplication of subspaces is the same as in the case where R is an alternative ring*; i.e., *we have the following table which is to be read as the table in Lemma 6*:

	R_{11}	R_{10}	R_{01}	R_{00}
R_{11}	R_{11}	R_{10}	0	0
R_{10}	0	R_{01}	R_{11}	R_{10}
R_{01}	R_{01}	R_{00}	R_{10}	0
R_{00}	0	0	R_{01}	R_{00}

Proof. In view of Lemmas 6, 10, and 12 it remains only to show that $R_{ii}R_{ii} \subset R_{ii}$. By Lemma 9, Eq. (11) we have $(y_{ij}, r_{ii}, s_{ii}) = 0$; thus $y_{ij} \cdot r_{ii}s_{ii} = -(y_{ij}, r_{ii}, s_{ii}) = 0$ since the other term in the expansion of the associator vanishes by Lemma 6. Also by Lemma 6 we have $r_{ii}s_{ii}$ is in $R_{ii} + R_{ji}$. Let $r_{ii}s_{ii} = x_{ii} + w_{ji}$. Then we have $0 = y_{ij} \cdot r_{ii}s_{ii} = y_{ij}x_{ii} + y_{ij}w_{ji} = y_{ij}w_{ji}$. Since y_{ij} is arbitrary in R_{ij} we conclude that w_{ji} is in S_{ji}; hence $w_{ji} = 0$, and thus $r_{ii}s_{ii}$ is an element of R_{ii}. This completes the proof of the lemma.

We proceed to show that, under the hypotheses of Lemmas 12 and 13, R is actually alternative. First we notice that since R_{11} and R_{00} are subrings by Lemma 13 and since they contain no nilpotent elements by assumption we may appeal to a result of Kleinfeld [*6*] to conclude that R_{11} and R_{00} are both alternative rings. In Lemmas 14–17 we assume R is as in Lemma 12.

LEMMA 14.

$$(\sigma(R_{ij}), \sigma(R_{ii}), \sigma(R_{ii})) = 0, \tag{15}$$

$$(\sigma(R_{ji}), \sigma(R_{ii}), \sigma(R_{ij})) = 0, \tag{16}$$

$$(\sigma(R_{ii}), \sigma(R_{jj}), \sigma(R_{ij})) = 0, \tag{17}$$

$$(\sigma(R_{ii}), \sigma(R_{ii}), \sigma(R_{jj})) = 0, \tag{18}$$

where σ indicates a permutation of the three entries within the associator.

Proof. This lemma follows immediately from the right alternative law (1), Lemma 13, and the definition of the associator.

LEMMA 15. *The elements x_{ii}, y_{ij}, and z_{ij} form an alternative triple, and so do the elements x_{ij}, y_{ij}, and z_{ji}.*

Proof. Using Lemmas 13 and 7 we have $(y_{ij}, z_{ij}, x_{ii}) = y_{ij}z_{ij} \cdot x_{ii} = -z_{ij}y_{ij} \cdot x_{ii}$. On the other hand, we have $-(z_{ij}, y_{ij}, x_{ii}) = -z_{ij}y_{ij} \cdot x_{ii}$

because of Lemma 13. Comparing these equations gives $(y_{ij}, z_{ij}, x_{ii}) = -(z_{ij}, y_{ij}, x_{ii})$. Similarly, we have $(x_{ii}, y_{ij}, z_{ij}) = -(x_{ii}, z_{ij}, y_{ij}) = -x_{ii}z_{ij} \cdot y_{ij} = y_{ij} \cdot x_{ii}z_{ij}$, while expanding and using Lemma 13 gives $-(y_{ij}, x_{ii}, z_{ij}) = y_{ij} \cdot x_{ii}z_{ij}$. Comparing the two preceding equations gives $(x_{ii}, y_{ij}, z_{ij}) = -(y_{ij}, x_{ii}, z_{ij})$. The preceding is sufficient to show that x_{ii}, y_{ij}, and z_{ij} form an alternative triple. Similarly, $(x_{ij}, y_{ij}, z_{ji}) = x_{ij}y_{ij} \cdot z_{ji} = -y_{ij}x_{ij} \cdot z_{ji}$ because of Lemmas 13 and 7. Expanding gives $-(y_{ij}, x_{ij}, z_{ji}) = -y_{ij}x_{ij} \cdot z_{ji}$ and comparing this with the previous equation gives $(x_{ij}, y_{ij}, z_{ji}) = -(y_{ij}, x_{ij}, z_{ji})$. Again using Lemmas 13 and 7 we have $(x_{ij}, z_{ji}, y_{ij}) = -(x_{ij}, y_{ij}, z_{ji}) = -x_{ij}y_{ij} \cdot z_{ji} = z_{ji} \cdot x_{ij}y_{ij}$ and expanding gives $-(z_{ji}, x_{ij}, y_{ij}) = z_{ji} \cdot x_{ij}y_{ij}$. Comparison yields $(x_{ij}, z_{ji}, y_{ij}) = -(z_{ji}, x_{ij}, y_{ij})$.

LEMMA 16. $(\sigma(R_{ji}), \sigma(R_{ii}), \sigma(R_{ii})) = 0$.

Proof. That $(R_{ii}, R_{ii}, R_{ji}) = 0$ can be seen directly from Lemma 13; thus by (1) we have $(R_{ii}, R_{ji}, R_{ii}) = 0$ also. We have $0 = g(x_{ji}, w_{ij}, y_{ii}, z_{ii}) = (x_{ji}, w_{ij}, y_{ii}z_{ii}) + (x_{ji}, y_{ii}, w_{ij}z_{ii}) - (x_{ji}, w_{ij}, z_{ii})y_{ii} - (x_{ji}, y_{ii}, z_{ii})w_{ij}$. Now $w_{ij}z_{ii} = 0$ by Lemma 13, so the second associator in the sum vanishes. Moreover by Eq. (16) of Lemma 14 the first and third associators vanish also. Thus $(x_{ji}, y_{ii}, z_{ii})w_{ij} = 0$. Expanding and using Lemma 13 we see that (x_{ji}, y_{ii}, z_{ii}) belongs to R_{ji}, and since w_{ij} is arbitrary in R_{ij} we have that (x_{ji}, y_{ii}, z_{ii}) is in S (as defined in Theorem 1) and thus $(x_{ji}, y_{ii}, z_{ii}) = 0$.

LEMMA 17. *The elements x_{ji}, y_{ji}, and z_{ii} form an alternative triple.*

Proof. Using Lemmas 13 and 7 we have $(x_{ji}, y_{ji}, z_{ii}) = -x_{ji} \cdot y_{ji}z_{ii} = y_{ji}z_{ii} \cdot x_{ji}$. However we also have $-(y_{ji}, x_{ji}, z_{ii}) = (y_{ji}, z_{ii}, x_{ji}) = y_{ji}z_{ii} \cdot x_{ji}$. Comparison yields $(x_{ji}, y_{ji}, z_{ii}) = -(y_{ji}, x_{ji}, z_{ii})$. Since R has characteristic different from 2, the previous equation implies $(x_{ji}, x_{ji}, z_{ii}) = 0$, and since $x_{ji}^2 = 0$ this gives $x_{ji} \cdot x_{ji}z_{ii} = 0$. By Lemma 11 we have $t_{ji}(t_{ji}z_{ii} \cdot y_{ji}) = y_{ji}(t_{ji} \cdot t_{ji}z_{ii}) = 0$. Substituting $x_{ji} + r_{ji}$ for t_{ji} in the previous equation yields

$$x_{ji}(r_{ji}z_{ii} \cdot y_{ji}) + r_{ji}(x_{ji}z_{ii} \cdot y_{ji}) = 0. \tag{19}$$

Now let us consider

$$r_{ji}[(z_{ii}, x_{ji}, y_{ji}) + (x_{ji}, z_{ii}, y_{ji})] = -r_{ji}(z_{ii} \cdot x_{ji}y_{ji}) + r_{ji}(x_{ji}z_{ii} \cdot y_{ji}). \tag{20}$$

By Eq. (16) we know that $(r_{ji}, z_{ii}, x_{ji}y_{ji}) = 0$. This and Lemma 11 imply $-r_{ji}(z_{ii} \cdot x_{ji}y_{ji}) = -r_{ji}z_{ii} \cdot x_{ji}y_{ji} = x_{ji}(r_{ji}z_{ii} \cdot y_{ji})$. Substituting this in (20) and comparing with (19) gives $r_{ji}[(z_{ii}, x_{ji}, y_{ji}) + (x_{ji}, z_{ii}, y_{ji})] = x_{ji}(r_{ji}z_{ii} \cdot y_{ji}) + r_{ji}(x_{ji}z_{ii} \cdot y_{ji}) = 0$. By Lemma 13 we have that

$(z_{ii}, x_{ji}, y_{ji}) + (x_{ji}, z_{ii}, y_{ji})$ is in R_{ij} and thus the preceding equation implies that this sum of associators is in S and hence is zero. This completes the proof of the lemma.

Since we have checked all possible ways of filling in the entries of an associator from the four subspaces and found that all obey the alternative law (see Lemmas 7, 14, 16, 17, and the observation following Lemma 13) we have proved the following theorem.

THEOREM 2. *A right alternative ring R of characteristic not 2 or 3 which has no proper nilpotent ideals and which has an idempotent e such that there are no nilpotent elements in $R_1(e)$ and $R_0(e)$ is alternative.*

Since a prime ring has no proper nilpotent ideals, we may replace the assumption that R has no nilpotent ideals in Theorem 2 by the assumption that R is prime, and the theorem remains true. By combining this with a result of Slater [*8*], which asserts that a prime alternative ring R of characteristic not 3 that is not associative can be embedded in a Cayley–Dickson algebra over the quotient field of the center of R, we can state the following theorem.

THEOREM 3. *A prime, right alternative ring R of characteristic not 2 or 3, with an idempotent e such that there are no nilpotent elements in $R_1(e)$ or $R_0(e)$ which is not associative, may be embedded in a Cayley–Dickson algebra over the quotient field of the center of R.*

Our work can also be applied when R is a simple right alternative ring. The following theorem gives a necessary and sufficient condition for a simple right alternative ring of characteristic not 2 or 3 which is not associative to be alternative.

THEOREM 4. *A simple right alternative ring R of characteristic not 2 or 3 which is not associative is alternative if and only if R has an idempotent e such that there are no nilpotent elements in $R_1(e)$ and $R_0(e)$.*

Proof. The direct part of the theorem follows from Theorem 2 since a simple ring has no nilpotent ideals. The indirect part of the theorem follows from the fact that the only simple alternative rings of characteristic not 2 or 3 which are not associative are the Cayley–Dickson division rings and the split Cayley–Dickson algebras [*5*], both of which are known to have an idempotent e such that there are no nilpotent elements in $R_1(e)$ and $R_0(e)$. In the case of the Cayley–Dickson division algebras, take $e = 1$. The split Cayley–Dickson algebra has an idempotent e such that $R_1(e)$ and $R_0(e)$ are both fields isomorphic to the center [*3*].

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